Theory of remaining energy
—wave mechanics of particles in curved space-time(Part III)

Yu Xianqiao[1,*]

[1]*School of Physical Science and Technology, Southwest University, Chongqing 400715, China*

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In this article, based on the wave equation established in the first part of this series, I have studied the evolution of the energy density of the universe from the initial moment to the present, and found that there is still residual energy of the initial energy of the universe, which appears in the form of vacuum energy. The residual energy is the simplest explanation for dark matter. The calculations show that the present average energy of each momentum mode in the residual energy is of the order of 10^{-3}eV, which is approximately 3 to 5 times the averaged energy of photons in the cosmic microwave background.

In this article, I will answer the questions raised at the end of the second part of this series [1]: why does the vacuum energy of the matter field make no contribution to the cosmological constant and why the current vacuum energy density is so small? I will first discuss the state of the universe at the initial moment. In modern cosmology, people's researches on the evolution of the universe can only begin after the Planck time. The period from the singularity($t = 0$) to the Planck time($t_P = 5.38 \times 10^{-44} s$) is called the Planck period. In the Planck period, due to the quantum fluctuations of space-time, the law of causality and all known physical laws are invalid, we call the matter in the Planck state in this period. The existing physical theory is powerless to describe the matter in Planck state. Therefore, according to the existing physical theory, our researches on the singularity of matter is limited beyond the Planck scale, and the researches on the evolution of the universe can only begin after the Planck period. However, now, the situation has changed and theoretical physics has made new progress. The universal wave equation that microscopic particles satisfy in any space-time scale in curved space-time is established [2]. Based on this new wave equation, we can scientifically describe the state of matter in the Planck period.

During the Planck period, microscopic particles satisfy the wave equation [2]

$$il_P\frac{\partial\psi}{\partial\tau} = -\frac{l_P^2}{2\sqrt{-g}}\Box^2(\sqrt{-g}\psi) - \frac{1}{2}\psi, \tag{1}$$

where l_P is Planck length, $\Box^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial t^2}$, $g = |g_{\mu\nu}|$ is the determinant of the metric tensor. τ is called proper time and there is a relationship $d\tau^2 = -ds^2 = -g_{\mu\nu}dx^\mu dx^\nu$.

In the Planck period, due to the quantum fluctuations of space-time, which are caused by the quantum fluctuations of matter field, the metric tensor $g_{\mu\nu}$ of space-time will oscillate rapidly with time, so the value of $\sqrt{-g}$ in wave equation(1) will also oscillate rapidly. In the very short Planck period, the value of $\sqrt{-g}$ in wave equation(1) should take its average value during this period $\overline{\sqrt{-g}}$. Because the value of $\sqrt{-g}$ oscillates rapidly, the most reasonable assumption is that during the Planck period of the Big Bang, $\overline{\sqrt{-g}} = \mathrm{constant}$, that is, the average effect of the rapidly oscillating $\sqrt{-g}$ is a constant. As a result, in the Planck period, the wave equation(1) satisfied by particles can be simplified as

$$il_P\frac{\partial\psi}{\partial\tau} = -\frac{l_P^2}{2}\Box^2\psi - \frac{1}{2}\psi. \tag{2}$$

Equation(2) is also the wave equation satisfied by free particles in the flat space without gravity($\sqrt{-g} = 1$), so the state wave functions of free particles in the flat space without gravity are also the states of particles in the Planck period. From this, it can be deduced that at the big bang singularity, the particle is in a certain energy state, that is, in a stationary state. The distribution of particles in energy levels should minimize the total energy of the whole system, only in this way can the system be most stable according to the principle of minimum energy. There is only one possibility, that is, all particles are in its lowest energy state(the state of $E = \frac{1}{2}\sqrt{k^2 + m^2}$), i. e. vacuum state. On the other hand, according to the thoughts of Laozi's *Tao Te Ching*, at the beginning of all things, everything is the simplest and becomes complicated after evolution, the universe must have evolved from simple to complex. The most reasonable assumption is that at the initial moment($t = 0$ moment), there is only one field in the universe, which I call the original field X, all other matter fields, such as the electron field, neutrino field, quark fields, and so on in the standard model, are generated from the evolution of the original field, that is, all currently known matter fields have a single common origin—the original field. There is a similar idea of common origin in biology. In the 19th century, Darwin

*Electronic address:yuxq@swu.edu.cn

put forward the idea of species homology in his book *The Origin of Species*, that is, all living things on the Earth have a single common ancestor, they all evolved from the same primitive life. In biology, this idea is called hypothesis of common ancestry. Analogically, in physics, all matter has a single common origin, which can be called hypothesis of common origin of matter.

Based on the above two discussions, we can get that at the singularity of the Big Bang, the energy density of the universe must be

$$\rho(t=0) = g\int_0^{k_{\mathrm{GF}}(t=0)} \frac{d^3k}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2} + \rho_s, \tag{3}$$

where the factor $g = 2s+1$ is the spin degeneracy of the original field, $k_{\mathrm{GF}}(t=0)$ is generalized Fermi momentum at initial moment, whose value is determined by the particle number density of the original field at the initial moment(see discussions later). The first term on the right side in formula(3) is the contribution of pure matter field, expressed in the form of vacuum energy, and the second term is the contribution of pure space, expressed in the form of constant. ρ_s remains unchanged throughout the history of the universe. It is an inherent basic attribute of space itself, which I have discussed in the second part of this series [1].

The original field, which is beyond the standard model, may be a scalar field, or a spin 1/2 field, or a vector field, or other fields. If the original field is a scalar field, various scalar field models have been fully studied in relevant literatures, and the simplest one is to add a term to the Lagrange of the standard model [3]

$$L_S = \partial^\mu S\partial_\mu S - m^2 S^\dagger S - \lambda_S S^\dagger S H^\dagger H. \tag{4}$$

This model has a global U(1) symmetry. If the original field is a vector field, the popular model is an extension of the standard model by an additional $U(1)_X$ gauge symmetry and a complex scalar field $\Phi = (\phi_r + i\phi_i)/\sqrt{2}$, whose vacuum expectation value generates a mass of this U(1)'s vector field. The Lagrangian of the scalars and the new vector boson is [4]

$$L = D_\mu\Phi D^\mu\Phi - \frac{V_{\mu\nu}V^{\mu\nu}}{4} - V(\Phi, H), \tag{5}$$

where $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$, $D_\mu = \partial_\mu - ig_V V_\mu$ and

$$V(\Phi, H) = -\mu_\phi^2|\Phi|^2 - \mu^2|H|^2 + \lambda_\phi|\Phi|^4 + \lambda|H|^4 + \lambda_{H\phi}|\Phi|^2|H|^2. \tag{6}$$

Equation(4) and equation(5) are only two specific examples of original field models. Due to the lack of sufficient experimental data at present, what kind of field the original field is and what the actual original field model is can only be determined through future experiments. In this article, I only discuss the evolution of the expression(3) of the energy density of the universe from the perspective of pure energy, my conclusion, the theory of residual energy, is not affected by specific models, just as the theory of evolution proposed by Darwin in his work on the origin of species is not affected by specific evolutionary mechanisms.

The energy of the original field is stored in the form of vacuum energy(the first term on the right side of formula(3)), which is a natural result of the wave equation(2) satisfied by particles during the Planck period and the principle of minimum energy. The question now is: does this vacuum energy change with space expansion?

(1) For a long time, there has been a firm and unshakable belief in people's hearts that vacuum energy density is a constant, which does not dilute with the expansion of space, just as people firmly believed that species were invariant before Darwin proposed the theory of evolution in the 19th century. In static space, vacuum energy density is indeed a constant, but when space expands, can vacuum energy density still maintain a constant? In fact, we lack rigorous experimental evidences to support that vacuum energy density does not change with space expansion.

(2) For an ideal fermion system, such as a free electron gas, at the temperature of $T = 0$ K, the electrons are arranged from the lowest energy level up until all the electrons are arranged. At this time, the highest energy level is called Fermi energy level, the corresponding maximum energy ε_F is called Fermi energy, and the maximum momentum k_F is called Fermi momentum. Thus, at absolute zero temperature, the electron gas is arranged in such a way that all energy levels below the Fermi level are completely filled by particles; All energy levels above the Fermi level are empty. Assuming that at $T = 0$ K, N fermions are confined to a container with a volume of V, its energy density is

$$\rho = g\int_0^{k_{\mathrm{F}}} \frac{d^3k}{(2\pi)^3}\sqrt{k^2+m^2}, \tag{7}$$

where the factor g is the spin degeneracy of the fermion. By carefully comparing the first term on the right side of equation(3) with equation(7), we find that the energy density of a vacuum state is very similar to that of a fermion gas at absolute zero. The only difference is that in a vacuum state, each particle with a momentum of k mode is in the lowest energy state of $E = \frac{1}{2}\sqrt{k^2+m^2}$, while in a fermion system, each particle with a momentum of k mode is in an excited state of $E = \sqrt{k^2+m^2}$. Based on this similarity, I refer to the maximum momentum in vacuum energy as generalized Fermi momentum, and the corresponding maximum energy as generalized Fermi energy. Therefore, we have sufficient reasons to conclude that the behavior

of an ideal Fermi gas at absolute zero temperature is very similar to the behavior of vacuum energy. We now examine whether equation(7) changes with volume expansion and its changing rules.

We calculate the integral in formula(7), and obtain

$$\begin{aligned}\rho &= g\int_0^{k_{\mathrm{F}}}\frac{d^3k}{(2\pi)^3}\sqrt{k^2+m^2}\\ &= \frac{gk_F^4}{8\pi^2}f\left(\frac{m}{k_{\mathrm{F}}}\right)\\ &= \frac{gk_{\mathrm{F}}^4}{8\pi^2}\left(1+\frac{m^2}{k_{\mathrm{F}}^2}+\cdots\right),\end{aligned} \tag{8}$$

where $f(x)=\sqrt{1+x^2}(1+\frac{1}{2}x^2)-\frac{1}{2}x^4\ln(\frac{1}{x}+\frac{1}{x}\sqrt{1+x^2})$ and in the last expression we have expanded the exact expression in terms of the small parameter m/k_{F}. Let's discuss it in two situations.

(A) Extreme relativity case. In this case, $m \ll k_{\mathrm{F}}$, formula(8) becomes

$$\rho = \frac{gk_F^4}{8\pi^2}, \tag{9}$$

where the Fermi momentum k_{F} is determined by the total particle number N and volume V

$$\begin{aligned}N &= g\int_0^{k_{\mathrm{F}}}\frac{d^3x d^3k}{(2\pi)^3}\\ &= \frac{gV}{6\pi^2}k_F^3\end{aligned} \tag{10}$$

From this, we obtain the Fermi momentum

$$k_F = \left(\frac{6\pi^2}{g}\right)^{1/3}\left(\frac{N}{V}\right)^{1/3} \tag{11}$$

As it can be seen from formula(11), for a fixed number of Fermion particles N, when the volume expands, the Fermi momentum decreases, and then according to formula(9), its energy density also decreases. Combining formula(9) and formula(11), it can be obtained that in the extreme relativistic case, the change rule of the energy density of an ideal Fermion gas at absolute zero temperature is

$$\rho V^{4/3} = Const. \tag{12}$$

(B) Non-relativistic case. In this case, $\sqrt{k^2+m^2}\approx\frac{k^2}{2m}+m$, and that $\frac{k^2}{2m}\ll m$, so formula(7) becomes

$$\begin{aligned}\rho &= g\int_0^{k_{\mathrm{F}}}\frac{d^3k}{(2\pi)^3}\sqrt{k^2+m^2}\\ &\simeq g\int_0^{k_{\mathrm{F}}}\frac{d^3k}{(2\pi)^3}m\\ &= \frac{gm}{6\pi^2}k_{\mathrm{F}}^3\end{aligned} \tag{13}$$

Combining formula(13) and formula(11), it can be obtained that in the non-relativistic case, the change rule of the energy density of an ideal Fermion gas at absolute zero temperature is

$$\rho V = Const. \tag{14}$$

Formula(12) and formula(14) are the change laws of the energy density of an ideal Fermi gas at absolute zero temperature under extreme relativistic and non relativistic conditions, respectively. Considering the close similarity between the vacuum energy density formula and the energy density formula for an ideal Fermi gas at absolute zero temperature, we have sufficient reasons to conclude that formula(12) and formula(14) are also the change laws of vacuum energy density in extreme relativistic and non relativistic situations, respectively. i. e.

$$\begin{cases}\rho_{\mathrm{vac}}V^{4/3} = Const. \ \ \text{Extreme relativity case}\\ \rho_{\mathrm{vac}}V = Const. \ \ \mathrm{Non-relativistic\ case}\end{cases} \tag{15}$$

Therefore, we can see that in a static space, the volume V is constant, and the vacuum energy density $\rho_{\rm vac}$ is indeed a constant. However, when the volume increases due to expansion, $\rho_{\rm vac}$ will decrease. In an expanding universe, $V \propto a^3(t)$, so formula(15) can be rewritten as

$$\begin{cases} \rho_{\rm vac} a^4(t) = Const. \ \ \text{Extreme relativity case} \\ \rho_{\rm vac} a^3(t) = Const. \ \ \text{Non}-\text{relativistic case} \end{cases} \tag{16}$$

where $a(t)$ is the scale factor of the expanding universe. This will enable us to answer the first question raised at the beginning of this article. In an expanding universe, the scale factor will continue to increase, according to formula(16), the vacuum energy density $\rho_{\rm vac}$ will continue to decrease and cannot be maintained as a constant. Therefore, the vacuum energy of matter field does not contribute to the cosmological constant. At the same time, from formula(16), we can also see that as the universe expands, the generalized Fermi momentum will continuously decrease, and the original field must undergo a change process from an extreme relativistic situation to a non -relativistic situation. As a result, the behavior of vacuum energy will also undergo a change process from the behavior of radiation-like to the behavior of non-relativistic matter-like, which is consistent with the analysis of the evolution of cosmic energy density in modern cosmology, the universe was dominated by radiation in its early stages, and later by matter.

Now I begin to discuss the evolution of the initial cosmic energy density formula(3). The first term on the right side of formula(3) is the vacuum energy of the original field, if the number of particles of the original field remains constant, its evolution will follow the law of formula(16). However, the actual evolution is slightly more complex than the law of formula(16), due to the release of vacuum energy to produce other particles. First, a definition is defined

Definition: If a vacuum state, real particle-antiparticle pairs can not be created spontaneously from it, we call it stable vacuum state; otherwise, it is unstable vacuum state.

At the initial moment of the universe, the vacuum represented on the right side of formula(3) must be unstable. Although each momentum mode k(with a energy $\frac{1}{2}\sqrt{k^2+m^2}$) in the vacuum energy is stable, two modes, k_1(with a energy $\frac{1}{2}\sqrt{k_1^2+m^2}$) and k_2(with a energy $\frac{1}{2}\sqrt{k_2^2+m^2}$), may annihilate into a pair of particles when they collide, as described by

$$k_1 + k_2 \rightarrow \text{particle} + \text{antiparticle}, \tag{17}$$

which explains the fact that elementary particles are produced from vacuum state in the Big Bang epoch of the universe. The process $k_1 + k_2 \rightarrow \gamma + \gamma$(just like $\nu + \bar{\nu} \rightarrow \gamma + \gamma$) is forbidden or strongly suppressed because of the fact that the original field is neutral, and thus is not involved in electromagnetic interaction. It must be emphasized that this process(17) can only occur when the energy provided($\frac{1}{2}\sqrt{k_1^2+m^2} + \frac{1}{2}\sqrt{k_2^2+m^2}$) exceeds the total mass of the resulting particles and antiparticles.

One may wonder whether vacuum energy can annihilate into elementary particles, as suggested in formula(17). By definition, vacuum state describes the lowest energy state in quantum field theory, which is indeed the case. The minimum energy($\frac{1}{2}\sqrt{k^2+m^2}$) of each momentum mode k is indeed smaller than the energy of the excited state particle($\sqrt{k^2+m^2}$), this means that each momentum mode k in the vacuum is stable and cannot decay. However, the fact is that one particle cannot spontaneously decay does not mean that two particles cannot annihilate. Just as one photon is stable and cannot decay into any particles, but two photons can annihilate(for example $\gamma+\gamma \rightarrow e^+ + e^-$), the same is true for each momentum mode k in vacuum energy.

What I want to emphasize is that I discuss the occurrence of reaction(17) from the perspective of pure energy conservation and mass energy relations. As long as the energy conditions are satisfied, reaction(17) will definitely occur. As for the specific value of the annihilation cross section of reaction(17) and how the elementary particles are generated from the original field, which is unknown at present due to the lack of experimental data, has no influences on our discussions and conclusions, fortunately. Then these produced elementary particles decayed or annihilated again, and leaving only photons, neutrinos, and light elements after the Big Bang Nucleosynthesis, in addition to the vacuum energy of the original field and the energy of space itself. The energy density of the universe at initial moment described by Eq.(3) becomes correspondingly

$$\rho(t) = \rho_\gamma(t) + \rho_\nu(t) + \rho_b(t) + g\int_0^{k_{\rm GF}(t)} \frac{d^3k}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2} + \rho_s. \tag{18}$$

The fourth item on the right side of Eq.(18) is the energy of the original field, expressed in the form of vacuum energy

$$\rho_{\rm vac}(t) = g\int_0^{k_{\rm GF}(t)} \frac{d^3k}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2} = \frac{g k_{\rm GF}^4(t)}{16\pi^2} f\left(\frac{m}{k_{\rm GF}(t)}\right), \tag{19}$$

which decreases with time for two reasons: First, vacuum energy is released to produce particles; Second, the expansion of the universe, the generalized Fermi momentum also decreases accordingly.

Assuming that at the moment $t = t_1$, $\frac{1}{2}\sqrt{k_{\rm GF}^2(t_1)+m^2} + \frac{1}{2}\sqrt{k_{\rm GF}^2(t_1)+m^2} \le 2m_\nu$, i.e. $k_{\rm GF}(t_1) \le 2m_\nu$, even the lightest neutrinos cannot spontaneously arise from a vacuum state, the vacuum becomes a stable vacuum. Some people may argue that it is precisely at this time, such as $t = t_2$, when the rate of spontaneous production of neutrinos and antineutrinos from a vacuum state becomes less than the expansion rate of the universe, the vacuum becomes a stable vacuum. The exact time t_2 is very difficult to determine because of the unknown annihilation cross section, but it does not change the fact that the vacuum was stable at the moment $t = t_1$ and our following calculations will not be affected by the unknown time t_2. When the vacuum becomes stable, the original field will also change from the initial extreme relativistic situation to a non-relativistic situation, and the vacuum energy at this time is referred to as the remaining vacuum energy. According to formula(16), we have $\rho_{\rm rvac}(t_2)a^3(t_2) = \rho_{\rm rvac}(t_1)a^3(t_1) = \rho_{\rm rvac}(t_0)a^3(t_0)$, here t_0 means the time present. The precise value of t_2 and $\rho_{\rm rvac}(t_2)$ are vague and unclear, but it doesn't matter. We can calculate the value of t_1 and $\rho_{\rm rvac}(t_1)$, which are only depend on neutrino's mass, and thus the calculation of the relic density $\rho_{\rm rvac}(t_0)$ is not be affected by the unknown annihilation cross section.

Therefore, we can get a picture of the universe's evolution: in addition to the very small energy density of space itself ρ_s, the initial energy of the universe is stored in the form of vacuum energy of a neutral original field. Because this vacuum state is unstable, as the universe expands, the original vacuum energy is divided into two parts. One part had been released and transformed into ordinary matter or radiation through formula(17), the other part remain in the form of vacuum energy. The part left I call it the remaining energy of the original field, whose energy density evolutes via $\rho_{\rm rvac}(t)a^3(t) = Const.$

When $k_{\rm GF}$ decreased to $2m_\nu$, at that moment I set the time $t = t_1$, the temperature $T = T_1$ and the scale factor $a = a_1$, according to Eq.(19), we have

$$\begin{aligned}\rho_{\rm rvac}(t_1) &= \frac{g(2m_\nu)^4}{16\pi^2} f\left(\frac{m}{2m_\nu}\right) \\ &= \frac{g m_\nu^4}{\pi^2} f\left(\frac{m}{2m_\nu}\right).\end{aligned} \tag{20}$$

Because $\rho_{\rm rvac}(t_1)a_1^3 = \rho_{\rm rvac}(t_0)a_0^3$, here t_0 is the present time and a_0 is the present value of cosmic scale factor, then we obtain the present remaining vacuum energy density

$$\begin{aligned}\rho_{\rm rvac}(t_0) &= \rho_{\rm rvac}(t_1)\left(\frac{a_1}{a_0}\right)^3 \\ &= \rho_{\rm rvac}(t_1)\left(\frac{T_0}{T_1}\right)^3 \\ &= \frac{g m_\nu^4}{\pi^2}\left(\frac{T_0}{T_1}\right)^3 f\left(\frac{m}{2m_\nu}\right),\end{aligned} \tag{21}$$

where T_0 is present cosmic plasma temperature, the second step is based on the formula that $a(t)T(t)$ remains unchanged during the evolution of the universe. Therefore, the ratio of the present remaining vacuum energy density to the critical density is

$$\begin{aligned}\Omega_{\rm rvac} &\equiv \frac{\rho_{\rm rvac}(t_0)}{\rho_{cr}} \\ &= \frac{g m_\nu^4}{\pi^2 \rho_{cr}}\left(\frac{T_0}{T_1}\right)^3 f\left(\frac{m}{2m_\nu}\right).\end{aligned} \tag{22}$$

Because at the temperature $T = T_1$, the lightest neutrinos can not be created spontaneously from the vacuum state, we must have $T_1 \le m_\nu$; on the other hand, approximately, $T_1 > 0.1m_\nu$(the reason will be given in the appendix). It is reasonable to take the temperature T_1 in the range of $0.1m_\nu < T_1 < m_\nu$. If we set $\Omega_{\rm rvac} = 0.26 \pm 0.01$, then we get a limit on the lightest neutrino mass m_ν, see Table I, Table II and Table III, where we have considered three cases: the original field is a scalar field, a spin-1/2 field and a vector field.

Here we have used the input parameters [5]: $\rho_{cr} = 1.879\, h^2 \times 10^{-29}{\rm g\ cm^{-3}} = 8.098\, h^2 \times 10^{-11}{\rm eV}^4$, $h = 0.72$, $T_0 = 2.725{\rm K} = 2.348 \times 10^{-4}{\rm eV}$. For the function $f(\frac{m}{2m_\nu})$, we take the value $f(\frac{m}{2m_\nu}) = 1.08^{+0.13}_{-0.07}$, the reason will be given later in the discussion on the equation of state. From Table I, Table II and Table III, we can see that the lightest neutrino mass is in the range of

$$0.008^{-0.002}_{+0.001}\ {\rm eV} \le m_\nu \le 7.698^{-1.091}_{+0.850}\ {\rm eV}, \tag{23}$$

if the original field is a scalar field; and

$$0.004^{-0.001}_{+0.000}\ {\rm eV} \le m_\nu \le 3.849^{-0.546}_{+0.425}\ {\rm eV}, \tag{24}$$

TABLE I. The limit on the lightest neutrino mass m_ν(in unit of eV) from the present remaining vacuum energy density if the original field is a scalar field, the error comes from the function $f(\frac{m}{2m_\nu})$.

$m_\nu \backslash T_1$ / $\Omega_{\rm rvac}$	$0.1m_\nu$	$0.2m_\nu$	$0.3m_\nu$	$0.4m_\nu$	$0.5m_\nu$	$0.6m_\nu$	$0.7m_\nu$	$0.8m_\nu$	$0.9m_\nu$	m_ν
0.25	$0.007^{-0.001}_{+0.001}$	$0.059^{-0.006}_{+0.004}$	$0.200^{-0.021}_{+0.014}$	$0.474^{-0.051}_{+0.033}$	$0.925^{-0.099}_{+0.064}$	$1.599^{-0.172}_{+0.111}$	$2.539^{-0.273}_{+0.176}$	$3.790^{-0.407}_{+0.263}$	$5.396^{-0.580}_{+0.374}$	$7.402^{-0.795}_{+0.513}$
0.26	$0.008^{-0.001}_{+0.001}$	$0.062^{-0.007}_{+0.004}$	$0.208^{-0.022}_{+0.014}$	$0.493^{-0.053}_{+0.034}$	$0.962^{-0.103}_{+0.067}$	$1.663^{-0.179}_{+0.115}$	$2.640^{-0.284}_{+0.183}$	$3.941^{-0.423}_{+0.273}$	$5.612^{-0.603}_{+0.389}$	$7.698^{-0.827}_{+0.534}$
0.27	$0.008^{-0.001}_{+0.001}$	$0.064^{-0.007}_{+0.005}$	$0.216^{-0.023}_{+0.015}$	$0.512^{-0.055}_{+0.036}$	$0.999^{-0.107}_{+0.069}$	$1.727^{-0.186}_{+0.120}$	$2.742^{-0.295}_{+0.190}$	$4.093^{-0.440}_{+0.284}$	$5.827^{-0.626}_{+0.404}$	$7.994^{-0.859}_{+0.554}$

TABLE II. The limit on the lightest neutrino neutrino mass m_ν(in unit of eV) from the present remaining vacuum energy density if the original field is a spin-1/2 field, the error comes from the function $f(\frac{m}{2m_\nu})$.

$m_\nu \backslash T_1$ / $\Omega_{\rm rvac}$	$0.1m_\nu$	$0.2m_\nu$	$0.3m_\nu$	$0.4m_\nu$	$0.5m_\nu$	$0.6m_\nu$	$0.7m_\nu$	$0.8m_\nu$	$0.9m_\nu$	m_ν
0.25	$0.004^{-0.001}_{+0.000}$	$0.030^{-0.004}_{+0.002}$	$0.100^{-0.011}_{+0.007}$	$0.237^{-0.026}_{+0.016}$	$0.463^{-0.050}_{+0.032}$	$0.799^{-0.086}_{+0.056}$	$1.269^{-0.136}_{+0.088}$	$1.895^{-0.204}_{+0.131}$	$2.698^{-0.290}_{+0.187}$	$3.701^{-0.398}_{+0.256}$
0.26	$0.004^{-0.001}_{+0.000}$	$0.031^{-0.004}_{+0.002}$	$0.104^{-0.011}_{+0.007}$	$0.246^{-0.026}_{+0.017}$	$0.481^{-0.052}_{+0.033}$	$0.831^{-0.089}_{+0.058}$	$1.320^{-0.142}_{+0.092}$	$1.971^{-0.212}_{+0.136}$	$2.806^{-0.302}_{+0.194}$	$3.849^{-0.414}_{+0.267}$
0.27	$0.004^{-0.000}_{+0.000}$	$0.032^{-0.003}_{+0.002}$	$0.108^{-0.012}_{+0.007}$	$0.256^{-0.028}_{+0.018}$	$0.500^{-0.054}_{+0.034}$	$0.863^{-0.092}_{+0.060}$	$1.371^{-0.147}_{+0.095}$	$2.046^{-0.219}_{+0.142}$	$2.914^{-0.313}_{+0.202}$	$3.997^{-0.430}_{+0.277}$

if the original field is a spin-1/2 field; and

$$0.003^{-0.001}_{+0.000}\ \text{eV} \le m_\nu \le 2.566^{-0.364}_{+0.283}\ \text{eV}, \tag{25}$$

if the original field is a vector field, which are consistent with the latest upper limit on the absolute mass scale of neutrinos from Karlsruhe Tritium Neutrino experiment KATRIN [6]

$$m_\nu < 1.1\ \text{eV}(90\%\text{C.L.}). \tag{26}$$

and the recent upper bound for the lightest neutrino mass from data of the large scale structure of galaxies, cosmic microwave background, type Ia supernovae, and big bang nucleosynthesis [7]

$$m_\nu < 0.086\ \text{eV}(95\%\text{C.L.}). \tag{27}$$

Formulas(23), (24) and (25) also implies that the sum of the three generation neutrinos masses $\sum m_{\nu_i} \ge 0.024^{-0.006}_{+0.003}$ eV or $\sum m_{\nu_i} \ge 0.012^{-0.003}_{+0.000}$ eV or $\sum m_{\nu_i} \ge 0.009^{-0.003}_{+0.000}$ eV, respectively corresponding to the case where the original field is a scalar field, a spin-1/2 field, and a vector field, which are consistent with the minimum sum of the masses derived from atmospheric and solar neutrino oscillation data [8, 9]

$$\sum m_{\nu_i} \ge 0.0584^{+0.0012}_{-0.0008}\ \text{eV}. \tag{28}$$

In this way, from formula(22), I can answer the second question raised at the beginning of this article: why is the current vacuum energy density so small? The reason is that as the universe continues to expand, the vacuum energy density will continue

TABLE III. The limit on the lightest neutrino mass m_ν(in unit of eV) from the present remaining vacuum energy density if the original field is a vector field, the error comes from the function $f(\frac{m}{2m_\nu})$.

$m_\nu \backslash T_1$ / $\Omega_{\rm rvac}$	$0.1m_\nu$	$0.2m_\nu$	$0.3m_\nu$	$0.4m_\nu$	$0.5m_\nu$	$0.6m_\nu$	$0.7m_\nu$	$0.8m_\nu$	$0.9m_\nu$	m_ν
0.25	$0.002^{-0.000}_{+0.001}$	$0.020^{-0.002}_{+0.001}$	$0.067^{-0.008}_{+0.004}$	$0.158^{-0.017}_{+0.011}$	$0.308^{-0.033}_{+0.022}$	$0.533^{-0.057}_{+0.037}$	$0.846^{-0.091}_{+0.059}$	$1.263^{-0.136}_{+0.088}$	$1.799^{-0.194}_{+0.124}$	$2.467^{-0.265}_{+0.171}$
0.26	$0.003^{-0.001}_{+0.000}$	$0.021^{-0.003}_{+0.001}$	$0.069^{-0.007}_{+0.005}$	$0.164^{-0.017}_{+0.012}$	$0.321^{-0.035}_{+0.022}$	$0.554^{-0.059}_{+0.039}$	$0.880^{-0.094}_{+0.061}$	$1.314^{-0.141}_{+0.091}$	$1.871^{-0.201}_{+0.129}$	$2.566^{-0.276}_{+0.178}$
0.27	$0.003^{-0.001}_{+0.000}$	$0.021^{-0.002}_{+0.002}$	$0.072^{-0.008}_{+0.005}$	$0.171^{-0.019}_{+0.011}$	$0.333^{-0.036}_{+0.023}$	$0.576^{-0.062}_{+0.039}$	$0.914^{-0.098}_{+0.063}$	$1.364^{-0.146}_{+0.095}$	$1.942^{-0.208}_{+0.135}$	$2.665^{-0.287}_{+0.184}$

to decrease, and now the vacuum energy density has dropped to a very small value. Furthermore, according to formula(18), it can be obtained that

$$\rho_{\rm vac}(t_0) < \rho_{cr}(t_0) \sim 10^{-11}{\rm eV}^4, \tag{29}$$

which is consistent with the limitations on vacuum energy density from astronomical observations.

The whole vacuum energy of the original field has its own state equation, we rewrite Eq.(19) as

$$\begin{aligned}\rho_{\rm vac}(t) &= g\int_0^{k_{\rm GF}(t)} \frac{d^3k}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2} \\ &= \frac{gk_{\rm GF}^4(t)}{16\pi^2} f\left(\frac{m}{k_{\rm GF}(t)}\right) \\ &= \frac{gk_{\rm GF}^4(t)}{16\pi^2}\left(1+\frac{m^2}{k_{\rm GF}^2(t)}+\cdots\right),\end{aligned} \tag{30}$$

where in the last expression, we have expanded the exact expression in terms of the small parameter $m/k_{\rm GF}(t)$. The pressure of the vacuum energy reads [5, 10]

$$\begin{aligned}p_{\rm vac}(t) &= \frac{g}{6}\int_0^{k_{\rm GF}(t)} \frac{d^3k}{(2\pi)^3}\frac{k^2}{\sqrt{k^2+m^2}} \\ &= \frac{1}{3}\frac{gk_{\rm GF}^4(t)}{16\pi^2} h\left(\frac{m}{k_{\rm GF}(t)}\right) \\ &= \frac{1}{3}\frac{gk_{\rm GF}^4(t)}{16\pi^2}\left(1-\frac{m^2}{k_{\rm GF}^2(t)}+\cdots\right),\end{aligned} \tag{31}$$

where $h(x) = \sqrt{1+x^2}(1-\frac{3}{2}x^2)+\frac{3}{2}x^4\ln(\frac{1}{x}+\frac{1}{x}\sqrt{1+x^2})$. Let's discuss it in two situations.

(A) Extreme relativity case. In this case, $m \ll k_{\rm GF}(t)$, formula(31) becomes

$$p_{\rm vac}(t) \simeq \frac{1}{3}\rho_{\rm vac}(t) \tag{32}$$

(B) Non-relativistic case. In this case, $\sqrt{k^2+m^2} \approx \frac{k^2}{2m}+m$, and that $\frac{k^2}{2m} \ll m$, so formula(31) becomes

$$\begin{aligned}p_{\rm vac}(t) &= \frac{g}{6}\int_0^{k_{\rm GF}(t)} \frac{d^3k}{(2\pi)^3}\frac{k^2}{\sqrt{k^2+m^2}} \\ &\simeq 0\end{aligned} \tag{33}$$

So the equation of state for vacuum energy can be summarized as

$$\begin{cases} p_{\rm vac}(t) \simeq \frac{1}{3}\rho_{\rm vac}(t). & \text{Extreme relativity case} \\ p_{\rm vac}(t) \simeq 0. & \text{Non}-\text{relativistic case}\end{cases} \tag{34}$$

which is completely consistent with the previous analysis formula(16) on the behavior of vacuum energy. When $k_{\rm GF}(t) \leq 2m_\nu$, the vacuum became stable, numerical analysis shows that the equation of state for residual vacuum energy begins to deviate from the behavior of radiation when $\frac{m}{2m_\nu} > 0.1$, which gives a strong lower limit on the mass of the original field. In addition, if we assume that $m < m_\nu$, then the mass of the original field will be limited by $0.2m_\nu < m < m_\nu$, i. e. $0.1 < \frac{m}{2m_\nu} < 0.5$, so we get $f\left(\frac{m}{2m_\nu}\right) = 1.08^{+0.13}_{-0.07}$. Of course, it is also possible if $m > m_\nu$, but it will not differ by two orders of magnitude from the neutrino mass, otherwise the theoretical limit on the upper limit of neutrino mass will be smaller than the experimental upper bound(26)and (27).

To summarize briefly, the vacuum energy of the original field behaves like radiation in the early universe, and later behaves like non relativistic matter. There are two possibilities for interpreting this portion of residual vacuum energy.

(1) The first possibility. The residual energy of the original field is the dark matter that we have been struggling to find for a long time, this is the simplest explanation for dark matter.

(2) The second possibility. The residual energy of the original field is not dark matter, which is composed of other particles. Because dark matter particles can only be generated from the vacuum energy of the original field(the first term on the right side in Eq.(3)), Eq.(18) should be modified to

$$\rho(t) = \rho_\gamma(t)+\rho_\nu(t)+\rho_b(t)+\rho_{\rm DM}(t)+g\int_0^{k_{\rm GF}(t)}\frac{d^3k}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2}+\rho_s. \tag{35}$$

Regardless of which of the two possibilities, first, the residual vacuum energy of the original field must exist; Secondly, the second possibility is obviously more complex than the first, and does not conform to the principle of simplicity. According to Occam's razor principle, if it is not necessary, do not add entities, the simplest explanation for a phenomenon is often more accurate than the more complex explanation. These arguments and Occam's razor lead me to the conclusion that the relic of the original field which remains in the form of vacuum energy is the most likely candidate to be the main component of dark matter in the universe, I named it the residual energy theory of dark matter.

Some people may object that if dark matter is a relic of the original field, our limit on the mass of the original field is that its mass is of the order of neutrino mass or slightly larger, how is this consistent with structure formation constraints on warm or hot dark matter, that currently place a lower limit on a warm dark matter mass $m_{\rm DM} >$ a few keV? This seems to be contradictory. My answer is that in the residual energy theory of dark matter, the particles of the original field are always in the lowest energy state of $E = \frac{1}{2}\sqrt{k^2+m^2}$, its energy density is always in the form of vacuum energy

$$\rho_{\rm vac} = g\int_0^{k_{\rm GF}} \frac{d^3k}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2}. \tag{36}$$

However, in the studies of dark matter particles, regardless of whether it is a cold dark matter model or a hot dark matter model, dark matter particles are always in an excited state with an energy of $E = \sqrt{k^2+m^2}$, its energy density is

$$\rho_{\rm DM} = g\int_0^{\infty} \frac{d^3k}{(2\pi)^3} f(k)\sqrt{k^2+m^2}, \tag{37}$$

where $f(k)$ is the distribution function of dark matter particles. In the equilibrium state at temperature T, if dark matter is a boson, its distribution function is

$$f_{\rm BE} = \frac{1}{e^{(E-\mu)/T}-1}, \tag{38}$$

and if dark matter is a fermion, its distribution function is

$$f_{\rm FD} = \frac{1}{e^{(E-\mu)/T}+1}, \tag{39}$$

with μ the chemical potential. Formula(36) is obviously different from formula(37), so the same energy density constraints have different restrictions on the original field mass in formula(36) and on the dark matter particle mass in formula(37). In general, $f \ll 1$, let $\rho_{\rm vac} = \rho_{\rm DM}$, from formula(36) and formula(37), it can be seen that the original field mass in formula(36) is much smaller than the dark matter particle mass in formula(37). This explain the apparent contradiction. The current limit on the mass of warm dark matter particles, $m_{\rm DM} >$ a few keV, is given according to formula(37), while our limit on the mass of the original field is given according to formula(36), so the mass of the original field is much smaller if dark matter is the relic of original field.

Finally, I will calculate the average energy of each momentum mode k in the current remaining energy. The present residual energy density of the original field is

$$\begin{aligned}\rho_{\rm rvac}(t_0) &= \rho_{cr} \times \Omega_{\rm rvac} \\ &= \frac{g k_{\rm GF}^4(t_0)}{16\pi^2} f\left(\frac{m}{k_{\rm GF}(t_0)}\right),\end{aligned} \tag{40}$$

we set $\Omega_{\rm rvac} = 0.26 \pm 0.01$, then approximately we get the present generalized Fermi momentum is

$$k_{\rm GF}(t_0) = 0.005^{+0.000}_{-0.000}\ {\rm eV}, \tag{41}$$

for original vector field and original spin-1/2 field, and

$$k_{\rm GF}(t_0) = 0.006^{+0.001}_{-0.000}\ {\rm eV}, \tag{42}$$

for primordial scalar field. Then the present generalized Fermi energy is $\frac{1}{2}k_{\rm GF}(t_0) = 0.003^{+0.000}_{-0.000}$ eV(vector case and spin-1/2 case) or $\frac{1}{2}k_{\rm GF}(t_0) = 0.003^{+0.001}_{-0.000}$ eV(scalar case). The mode number density of remaining vacuum energy is

$$\begin{aligned}n_{\rm vac}(t) &= g\int_0^{k_{\rm GF}(t)} \frac{d^3k}{(2\pi)^3} \\ &= \frac{g k_{\rm GF}^3(t)}{6\pi^2},\end{aligned} \tag{43}$$

so its present value is $n_{\rm vac}(t_0) = 6.34\times10^{-9}{\rm eV}^3 \simeq 825483\ {\rm cm}^{-3}$ for original vector field or $n_{\rm vac}(t_0) = 4.23\times10^{-9}{\rm eV}^3 \simeq 550756\ {\rm cm}^{-3}$ for original spin-1/2 field or $n_{\rm vac}(t_0) = 3.65\times10^{-9}{\rm eV}^3 \simeq 475238\ {\rm cm}^{-3}$ for original scalar field, which are much larger than the number density of cosmic microwave background($n_\gamma \simeq 411{\rm cm}^{-3}$). Therefore we obtain the present averaged energy of each momentum mode in remaining vacuum energy

$$\begin{aligned}\overline{\frac{1}{2}k} &= \frac{\rho_{\rm rvac}(t_0)}{n_{\rm vac}(t_0)}\\ &= \frac{\Omega_{\rm rvac}\times\rho_{cr}}{n_{\rm vac}(t_0)}\\ &= (1.72^{+0.07}_{-0.06})\times10^{-3}{\rm eV},\end{aligned} \tag{44}$$

if the original field is a vector field; and

$$\begin{aligned}\overline{\frac{1}{2}k} &= \frac{\rho_{\rm rvac}(t_0)}{n_{\rm vac}(t_0)}\\ &= \frac{\Omega_{\rm rvac}\times\rho_{cr}}{n_{\rm vac}(t_0)}\\ &= (2.58^{+0.10}_{-0.10})\times10^{-3}{\rm eV},\end{aligned} \tag{45}$$

if the original field is a spin-1/2 field; and

$$\begin{aligned}\overline{\frac{1}{2}k} &= \frac{\rho_{\rm rvac}(t_0)}{n_{\rm vac}(t_0)}\\ &= \frac{\Omega_{\rm rvac}\times\rho_{cr}}{n_{\rm vac}(t_0)}\\ &= (2.99^{+0.12}_{-0.11})\times10^{-3}{\rm eV},\end{aligned} \tag{46}$$

if the original field is a scalar field, which are about three to five times the averaged energy of photons in cosmic microwave background.

Explanatory note: in this article, the symbol g I use has different meanings in different formulas. In the wave equation, such as Eq.(1), g represents the determinant of metric tensor; In the expression of energy density, such as formula(3) and formula(7), g represents the spin degeneracy. Readers should pay attention to distinction.

APPENDIX

In the appendix, I give the reason for $T_1 > 0.1m_\nu$.

From $\rho_{\rm rvac}(t)a^3(t) = Const$, we have

$$\frac{gk^4_{\rm GF}(t_1)}{16\pi^2} f\left(\frac{m}{2m_\nu}\right)a_1^3 = \frac{gk^4_{\rm GF}(t_0)}{16\pi^2} f\left(\frac{m}{k_{\rm GF}(t_0)}\right)a_0^3, \tag{47}$$

so the relation between $k_{\rm GF}(t_1)$ and $k_{\rm GF}(t_0)$ is

$$k_{\rm GF}(t_1) = a_1^{-3/4}k_{\rm GF}(t_0)\left[\frac{f\left(\frac{m}{k_{\rm GF}(t_0)}\right)}{f\left(\frac{m}{2m_\nu}\right)}\right]^{\frac{1}{4}}, \tag{48}$$

where we have used $a_0 = 1$. Therefore from Eq. (48) and the relation formula $T_1a_1 = T_0$, we obtain

$$\begin{aligned}\frac{T_1}{\frac{1}{2}k_{\rm GF}(t_1)} &= \frac{\frac{T_0}{a_1}}{\frac{1}{2}k_{\rm GF}(t_0)a_1^{-3/4}\left[\frac{f\left(\frac{m}{k_{\rm GF}(t_0)}\right)}{f\left(\frac{m}{2m_\nu}\right)}\right]^{\frac{1}{4}}}\\ &= a_1^{-1/4}\frac{T_0}{\frac{1}{2}k_{\rm GF}(t_0)}\left[\frac{f\left(\frac{m}{2m_\nu}\right)}{f\left(\frac{m}{k_{\rm GF}(t_0)}\right)}\right]^{\frac{1}{4}}.\end{aligned} \tag{49}$$

Eq. (41) and Eq. (42) gives $\frac{1}{2}k_{\rm GF}(t_0) \simeq 0.003$ eV. Numerical analysis shows that the value of $\left[\frac{f\left(\frac{m}{2m_\nu}\right)}{f\left(\frac{m}{k_{\rm GF}(t_0)}\right)}\right]^{\frac{1}{4}}$ is approximately equal to 1, and substituting the values $T_0 = 2.348 \times 10^{-4}$ eV, $\frac{1}{2}k_{\rm GF}(t_0) \simeq 0.003$ eV and $\frac{1}{2}k_{\rm GF}(t_1) = m_\nu$ into Eq. (49), we obtain

$$T_1 \simeq \frac{1}{a_1^4} \times (0.1 m_\nu). \tag{50}$$

Since $0 < a_1 < 1$, we get approximately $T_1 > 0.1 m_\nu$.

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